

Calc 3

D. M. Y
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Ex Find the points on the sphere $x^2 + y^2 + z^2 = 4$ closest to and furthest from $(3, -1, 1)$

↳ Trying to optimize distance subject to sphere

optimize $f(x, y, z) = (x-3)^2 + (y+1)^2 + (z-1)^2$
subject to $x^2 + y^2 + z^2 = 4$

$$f(x, y, z) = x^2 - 6x + 9 + y^2 + 2y + 1 + z^2 - 2z + 1$$

$$(x^2 + y^2 + z^2) = (-11) + 6x - 2y + 2z$$

So we will work with

$$F(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$$

$$2 - 2\lambda z$$

$$\nabla F = \langle -6 - 2\lambda x, 2 - 2\lambda y, 2 - 2\lambda z \rangle$$

$$(x^2 + y^2 + z^2 = 4)$$

So $\nabla F = \vec{0}$ iff

$$\begin{cases} -6 - 2\lambda x = 0 \\ 2 - 2\lambda y = 0 \\ 2 - 2\lambda z = 0 \end{cases}$$

Observe $\lambda = \pm \frac{\sqrt{11}}{2}$

$$x^2 + y^2 + z^2 = 4$$

$$\lambda^2(x^2 + y^2 + z^2) = \lambda^2 4$$

$$-3^2 + 1^2 - 1^2 = 4^2 \rightarrow (\text{substituting } (3, 2, 3))$$

Now, there is two cases (2 pts)

$$\lambda = \frac{+\sqrt{11}}{2}, \text{ solve for } x, y, z$$

$$A = \left\langle \frac{-6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}} \right\rangle$$

$$B = \left\langle \frac{+6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right\rangle$$

$$f(A) = 15 - 6 \frac{-6}{\sqrt{11}} + 2 \left(\frac{2}{\sqrt{11}} \right) + 2 \left(\frac{-2}{\sqrt{11}} \right)$$

$$= 15 + \frac{44}{\sqrt{11}}$$

$$f(B) = 15 - 6 \frac{+6}{\sqrt{11}} - \frac{4}{\sqrt{11}} + \frac{4}{\sqrt{11}}$$

$$= 15 - \frac{36}{\sqrt{11}}$$

$f(A)$ is max distance² and $f(B)$ is min distance²
by Lagrange Multipliers. A is furthest B is closest

Exercise: A box is to be built with surface area 12. What's the maximum volume?

Double Integrals (15.1)

Idea ~ We have functions of ~~even~~ several variables. What should it mean to integrate them?



In Calc I: Area under the curve / antiderivative

In Calc III: ~~Area~~ The definite integral of f over region R should be the "net volume" of f over R

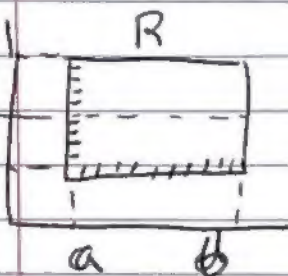


$$\iint_R f dA = \text{volume}$$

$\xrightarrow{\text{double}} \uparrow$ $\xrightarrow{\text{region}}$ $\xrightarrow{\text{respect to Area}}$

Today $\rightarrow R$ is as simple as possible (R is a rectangle)

Fixing y_0 creates a single var function $f(x, y_0)$



$$R = [a, b] [c, d] = \{(x, y) | x \in [a, b], y \in [c, d]\}$$

Using the same trick from Calc I, we can approx. $\int_R f$ the volume over R by using left lower corner points to determine height of boxes via $f(\text{corner point})$

Goal: use calculus I to solve $\iint_R f dA$

$$\iint_R f dA \rightarrow \int_x \int_y$$

Theo Prop ~ Fubini's Theorem: If $f(x, y)$ is
Cont on rectangle $R = [a, b] \times [c, d]$, then

$$\int_{y=c}^d \left(\int_{x=a}^b f(x, y) dx \right) dy = \iint_R f dA$$

(also works in other direction)

Ex. Compute $\iint_R x \sec^2(y) dA$ for $R = [0, 2] \times [0, \frac{\pi}{4}]$

By Fubini's Theorem

$$\int_{y=0}^{\frac{\pi}{4}} \left(\int_{x=0}^2 x \sec^2(y) dx \right) dy$$

$$= \int_{y=0}^{\frac{\pi}{4}} \sec^2(y) \left(\int_{x=0}^2 x dx \right) dy$$

$$= \int_{y=0}^{\frac{\pi}{4}} 2 \sec^2 y dy$$

$$= 2 \left[\tan y \right]_0^{\frac{\pi}{4}} = 2$$

ex. Compute $\iint_R \frac{1}{1+x+y} dA$ for $R = [1, 2] \times [2, 3]$

$$\int_{x=1}^2 \int_{y=2}^3 \frac{1}{1+x+y} dy dx \quad \begin{array}{l} u = 1+x+y \\ du = dy \end{array}$$

$$\int \frac{1}{u} du dx$$

$$\int_{x=1}^2 [\ln(14+x) - \ln(13+x)] dx$$

$$= (4+x) \ln(4+x) - (3+x) \ln(3+x) \Big|_{x=1}^2$$